DEVELOPMENT OF DYNAMIC FORMS OF STABILITY LOSS OF ELASTIC SYSTEMS UNDER INTENSIVE LOADING OVER A FINITE TIME INTERVAL

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Under the action of high intensity loads on elastic systems the largest deflectional growth rate is possessed by a characteristic form which differs from the first characteristic form [1]. For loads which exceed the Euler load the deflections of elastic systems tend towards infinity in the course of time [1, 2]. It is therefore natural to consider such problems over a finite time interval. We show below that in approximating an initial system with an infinite number of degrees of freedom over a finite time interval by a system with a finite number of degrees of freedom it is necessary to account for the density of the characteristic motions and information concerning external effects. We exhibit a formal process for identifying the fundamental motions in a problem involving the buckling of a rod. For the study of the motion over a sufficiently long time interval we can always replace a rod, acted on by a constant load of arbitrary intensity, by a system with a single degree of freedom if all the Fourier coefficients in the Fourier series for the perturbations are nonzero.

We give examples in which the density of the characteristic motions of elastic systems is sufficiently large, the density depending essentially on the thinness of the walls of a shell. We show that in a three-layered or multilayered assembly the calculation of transverse shear influences the density of the characteristic motions, this being in addition to the influence of the thinness of the walls of the structure. In all probability, both of these phenomena will be subject to further study.

We give recommendations for determining the critical time and the critical load intensity for systems with many degrees of freedom.

1. We consider the process of buckling of an elastic homogeneous rod under a constant intensity load

$$EIw_{,xxxx} + Nw_{,xx} + \rho F w_{,tt} = f(x) \quad (0 \leqslant x \leqslant L,\ t \geqslant 0) \tag{1.1}$$

Here w is the normal deflection; x and t are the longitudinal coordinate and the time; L is the length of the rod; ρ is the density of the rod material; F = const and I = const are the rod cross section and the rod bending rigidity; E is Young's modulus; N is a given constant longitudinal force; $f(x)$ is a function defining given perturbations or imperfections, assumed to be small [see Eqs. (1.11)].

We assume, up to the instant of loading, that the hinge-supported rod is at rest. Then the initial and boundary conditions have the form

$$w = w_{,t} = 0 \ \ (t = 0), \qquad w = w_{,xx} = 0 \qquad (x = 0,\ L) \tag{1.2}$$

We seek a solution of Eqs. (1.1) and (1.2) in the form

$$w = \sum_{m=1}^{\infty} q_m(t) \sin \frac{m\pi x}{L} \tag{1.3}$$

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After making appropriate transformations, we obtain equations for the $q_m(t)$ with zero initial conditions

$$q_m'' - \alpha_m^2 q_m = f_m, \quad q_m(0) = q_m'(0) = 0 \qquad (m = 1, 2, \ldots) \tag{1.4}$$

$$\alpha_m^2 = \frac{\pi^4 EI}{\rho F L^4} m^2 (\eta^2 - m^2), \quad \eta^2 = \frac{N}{N_e}, \quad N_e = \frac{\pi^2 EI}{L^2}, \quad f_m = \frac{2}{\rho F L} \int_0^L f(x) \sin \frac{m\pi x}{L} dx$$

Here $\eta \gg 1$ is a parameter characterizing the load intensity and N_e is the Euler load.

Indices $m < \eta$ correspond to motions describing a loss of stability

$$q_m = (f_m / \alpha_m^2)(\operatorname{ch} \alpha_m t - 1) \qquad (m < \eta) \tag{1.5}$$

For sufficiently large values of t the expressions (1.5) for q_m simplify:

$$q_m = (f_m / 2\alpha_m^2) \exp \alpha_m t \qquad (m < \eta) \tag{1.6}$$

Following [1], we single out from among the motions with indices $m < \eta$ that motion for which the coefficient in the exponent attains its largest value $\alpha^* = \alpha(m_*)$ (here m_* is the integer closest to $\eta/2^{1/2}$; the case in which $\alpha^* = \alpha(m_*) = \alpha(m_* + 1)$ is possible). It is obvious that as $t \to \infty$ we can replace a system with an infinite number of degrees of freedom by one of number m_* if $f(m_*) \neq 0$. We now point out a rule for choosing the degrees of freedom when the stability loss process is studied over a finite time interval $0 \leq t \leq t_0$. Consider the expressions

$$q_m^* = \frac{q_m(t_0)}{\operatorname{ch} \alpha^* t_0 - 1} = \frac{f_m}{\alpha_m^2} \frac{\operatorname{ch} \alpha_m t_0 - 1}{\operatorname{ch} \alpha^* t_0 - 1} \tag{1.7}$$

$$q_m^* = (f_m / \alpha_m^2) \exp (\alpha_m - \alpha^*) t_0 \tag{1.8}$$

Assume that we can divide the factors in the expressions (1.7) and (1.8) for q_m^* into classes at a specific time instant t_0:

$$\frac{\operatorname{ch} \alpha_m t_0 - 1}{\operatorname{ch} \alpha^* t_0 - 1} = O(\eta^{-i}), \quad m = m(i), \quad i = 0, 1, 2, \ldots, i_0 \tag{1.9}$$

$$\exp (\alpha_m - \alpha^*) t_0 = O(\eta^{-i}), \quad m = m(i), \quad i = 0, 1, 2, \ldots, i_0 \tag{1.10}$$

$$f_m / \alpha_m^2 = O(\eta^{-l}), \quad m = m(l), \quad l = 1, 2, \ldots, \infty \tag{1.11}$$

The first two relations depend on the time instant t_0 and on the density of the characteristic motions in the stability problem for a compressed rod, while the last relation is independent of the time [if some of the Fourier coefficients are zero, then for them $l = \infty$ in Eqs. (1.11)]. When $t_0 \to \infty$, in the first class $i = 0$ in Eq. (1.9) or (1.10) there is either one motion, for which $m = m_*$, or two motions, for which $m = m_*$, $m_* + 1$. Using Eqs. (1.9) and (1.11) or Eqs. (1.10) and (1.11), it is easy to identify the fundamental motions among the q_m^*. Actually, for q_m^* a comparison with respect to order holds at a fixed time instant:

$$q_m^*(t_0) = O(\eta^{-j}), \quad m = m(j), \quad j = j_0, \quad j_0 + 1, \ldots \tag{1.12}$$

Here j is an integral parameter. From the relations (1.12) we may determine the number of fundamental motions $m = m(j_0)$ ($\min j = j_0$). Among the fundamental motions the defining motion with index m_* is necessarily present for sufficiently large values of t_0 if $f_{m_*} \neq 0$.

Using the estimates (1.12), we drop all the secondary terms of the series (1.3), whereupon its principal part has the form

$$w(x, t) = \sum_m q_m(t) \sin \frac{m\pi x}{L}, \quad m = m(j_0) \tag{1.13}$$

For moderate times t_0 and external effects with a single order of smallness, the function $w(x, t_0)$ may be of a pulsating character with respect to the longitudinal coordinate.

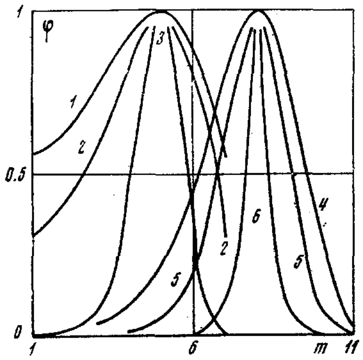

Fig. 1

We show that the separation into classes in Eqs. (1.9) and (1.10), and, correspondingly, also in Eqs. (1.12), depends on the rigidity characteristics of the rod; for example, for a three-layered rod with a supporting layer of small deflectional rigidity, the density of the characteristic motions for certain values of the load intensity increases substantially in comparison with the density of the characteristic motions for a homogeneous rod. For intensive loading of three-layered structures with a sufficiently pliable filler, higher forms of stability loss manifest themselves more readily than in homogeneous structures (see [2], pp. 806-809, and also [3, 4].

The function $\varphi(m, t_0, \eta) = \exp(\alpha_m - \alpha^*)$ is a distinct interior boundary layer [5] if we take the parameter m as the independent variable and fix t_0 and η. In Fig. 1 we show typical curves characterizing the distribution of amplitudes of stability loss forms relative to the determining motion at a fixed time instant when $f_m/\alpha_m^2 = \text{const} \neq 0$ for $m < \eta$. Curve 1 is drawn for $t = t_0$, curve 2 for $t = 2t_0$, curve 3 for $t = 10t_0$ ($\eta^2 = 50$), curve 4 for $t = t_0$, curve 5 for $t = 2t_0$, and curve 6 for $t = 10t_0$ ($\eta^2 = 128$). Under more intense loading, the boundary layer described by the function $\varphi(m, t_0, \eta)$ may be expressed more clearly. As $t \to \infty$ the function $\varphi(m, t_0, \eta)$ has the form $\varphi(m, \infty, \eta) = 1$ for $m = m_*$, $\varphi(m, \infty, \eta) = 0$ for $m \neq m_*$, i.e., as if it were completely "cut out" of a single motion [1, 5], if $f_{m*} \neq 0$.

In solving specific problems a separation into classes, similar to that in Eq. (1.9) or Eqs. (1.10) and (1.11), cannot always be made, since the boundaries of the classes may be "smeared out," but even in such cases identifying the fundamental motions presents no difficulties if use is made of a comparison with respect to order (asymptotic analysis).

Thus a rod under constant intensive loading may always be replaced by a system with a single degree of freedom of index $m = m_*$ [see Eq. (1.13)] if the time interval is sufficiently large ($0 \leq t \leq t_0$), and if the Fourier series coefficient of index m_* is nonzero, i.e., if $f_{m*} \neq 0$. In exceptional cases the rod may be replaced by a system with two degrees of freedom, with indices $m = m_*$ and $m_* + 1$, if t_0 is large and $f_{m*} \neq 0$, $f_{m*+1} \neq 0$.

The problem considered above, relating to the buckling process in a rod, is probably the simplest problem of its kind. We give below two examples which show that among the problems relating to the intensive loading of thin-walled elastic structures much more complicated situations may arise; in these examples the density of the characteristic motions depends essentially on the thin-walled property of the structure; moreover, in the second of these examples, the density of the characteristic motions is connected with the nonuniqueness of representation of the solution.

2. We now consider the development of axially symmetric forms of stability loss of thin-walled cylindrical shells under a constant intensive loading [2, 6]

$$\varepsilon^2 w_{,xxxx} + w + \lambda w_{,xx} + w_{,tt} = f(x) \qquad (0 \leqslant x \leqslant L/R,\ t \geqslant 0) \tag{2.1}$$

Here w is the normal deflection of the shell of length L and radius R; h is the shell thickness; x and t are, respectively, the longitudinal coordinate and the time; $\varepsilon^2 = h^2/12\,(1-\nu^2)\,R^2$ is a parameter which characterizes the thin-walled property of the structure; ν is Poisson's ratio; λ is a loading intensity parameter; and $f(x)$ is a function proportional to the given perturbations or imperfections of the shell (the scale of the independent variable t is chosen so that the coefficient of $w_{,tt}$ becomes unity).

Up to the instant of application of the intensive load, let the hinge-supported shell be at rest [see Eqs. (1.2), where x = 0, L/R]. If a solution of the problem (2.1), (1.2) is sought in the form of a series [see Eq. (1.3), where the independent variable is replaced by Rx], then for the q_m we obtain Eqs. (1.4), where the coefficients are of the form

$$\alpha_m^2 = \lambda m_1^2 - \varepsilon^2 m_1^4 - 1,\ m_1 = \frac{m\pi R}{L},\ f_m = \frac{2R}{L}\int_0^{L/R} f(x)\sin\frac{m\pi Rx}{L}\,dx \tag{2.2}$$

Next we study the motions describing loss of stability, the amplitudes of these motions being given by Eqs. (1.5) and (1.6) for $m_0 < m < m_{00}$; the indices $m < m_0$, $m > m_{00}$ correspond to oscillations; the time functions $q_{m_0}(t)$ and $q_{m_{00}}(t)$ may be second degree polynomials. The coefficient α_m has a maximum for

$$m_{1*}^2 = (m_* \pi R / L)^2 = \lambda / 2\varepsilon^2, \ \max \alpha_m^2 = \lambda^2 / 4\varepsilon^2 - 1 \tag{2.3}$$

We study the density of the characteristic motions. This problem is analogous to the problem concerning the number of characteristic values which fall in a given interval of variation of the characteristic values [7], the only difference being that in our problem it is necessary to count the number of characteristic motions (number of forms of stability) which fall in a given interval of variation of α_m [see Eqs. (1.7) and (1.9)]. From the expression for α_m^2 we have [see Eqs. (2.2)]

$$m_1 = \{\lambda/2\varepsilon^2 \pm [\lambda^2 - 4\varepsilon^2 (1 + \alpha_m^2)]^{1/2}/2\varepsilon^2\}^{1/2} \tag{2.4}$$

We establish the presence or absence of points (regions) of condensation for the function $m_1 = m_1(\alpha_m^2)$ by examining the derivative $\partial m_1/\partial \alpha_m^2$

$$\partial m_1/\partial \alpha_m^2 \sim [\lambda^2 - 4\varepsilon^2 (1 + \alpha_m^2)]^{-1/2} \tag{2.5}$$

We have omitted factors on the right side of the relation (2.5) which contain no singularities. From relation (2.5) it follows that $\partial m_1/\partial \alpha_m^2 \to \infty$ for $\lambda^2 \to 4\varepsilon^2 (1 + \alpha_m^2)$; in particular, when $\lambda^2 = 4\varepsilon^2 (1 + \alpha_m^2)$, the coefficient in the exponent attains a maximum value [see Eqs. (2.3)]. Thus in a neighborhood of $m = m_*$ we can find a region of condensation of characteristic motions. This region of condensation of characteristic motions influences the selection of the degrees of freedom when the stability loss process is studied over a finite time interval $0 \le t \le t_0$. Actually, the separation into the classes (1.9) and (1.10) depends on the density of the characteristic motions, the latter being determined by the thin-walled property of the structure. The function $\varphi(m, t_0, \eta, \varepsilon) = \exp(\alpha_m - \alpha^*) t_0$ is a distinctive interior boundary layer [5] (m is the independent variable; t_0, η, and ε are fixed).

The expression

$$w(x, t) = \sum_m q_m(t) \sin \frac{m\pi R}{4} x, \quad m = m(j_0, \varepsilon) \tag{2.6}$$

is the principal part of the series (1.3), wherein the number of terms in the sum (2.6) depends on the thin-walled property of the structure. In solving specific problems for a given thin-walled shell, the calculation of the density of the characteristic motions in a given problem presents no difficulties.

3. We consider the behavior of a spherical panel loaded by an intensive, external, uniformly distributed pressure. Let the fastening of the rectangular spherical panel admit a momentless state of stress and, at the instant of application of the large intensity load, let the panel be pressed without inertia. Then the equation describing the development of the normal deflections has the form [2, 6]

$$\varepsilon^2 \Delta\Delta w + w + \lambda \Delta w + w_{,tt} = f(x, y) \ \ (0 \leqslant x \leqslant a/R, \ \ 0 \leqslant y \leqslant b/R, \ \ t \geqslant 0) \tag{3.1}$$

Here x and y are spatial coordinates, R is the radius of the spherical shell, and the remaining notation is the same as that used in Sec. 1 ($\Delta w = w_{,xx} + w_{,yy}$).

We assume that prior to loading the hinge-supported shell is in a state of rest:

$$\begin{gathered} w = w_{,t} = 0, \qquad (t = 0), \quad w = w_{,xx} = 0 \\ (x = 0, \ a/R), \ w = w_{,yy} = 0 \quad (y = 0, \ b/R) \end{gathered} \tag{3.2}$$

We seek a solution of the Eqs. (3.1) and (3.2) in the form

$$w(x, y, t) = q(t) \, W(x, y) \tag{3.3}$$

where the function W(x, y) satisfies the equation (see [6], Chap. 10, § 15 and also [2], § 212)

$$\Delta W = -\mu^2 W \tag{3.4}$$

We obtain an equation for q(t), identical in form with Eqs. (2.4), which have already been studied. For fixed μ the Eq. (3.4) has the nonunique solution

$$\left(\frac{n\pi R}{a}\right)^2 + \left(\frac{m\pi R}{b}\right)^2 = \mu^2 \ \left(W(x, y) = \sin \frac{n\pi R x}{a} \sin \frac{m\pi R y}{b}\right) \tag{3.5}$$

either for integers n and m or for n and m differing little from integral values. In the first case the coefficients in the exponents of the corresponding motions coincide, while in the second case they differ little from each other. Thus the density of the characteristic motions in the spherical panel buckling problem depends on the thin-walled property of the structure and is connected with the nonuniqueness of the solution of Eq. (3.5).

For intensive loading of shells [2] the higher stability loss forms have the greatest amplitude growth rate; this was stated formally in [4]; however, it was not proved that, in all, a single form corresponds to the exponent with the maximum index. The example we have given shows that nonuniqueness may exist. Consequently, we cannot always succeed in approximating the initial system by one with a single degree of freedom. It is therefore desirable to conduct a supplementary study of the density of the characteristic motions in problems involving intense loading of arbitrary homogeneous and layered shells. Analogous problems concerning the density of the characteristic values in problems of dynamics [8] and stability [9] of homogeneous shells were studied earlier. Under intensive dynamic loading of shells the corresponding equations also include inertia terms and terms connected with stability.

Thus the principal part of the solution of the problem (3.1), (3.2), if an analysis similar to that of Sec. 1 is carried out, may be represented in the form

$$w = \sum_{n,m} q_{nm}(t) \sin \frac{n\pi Rx}{a} \sin \frac{m\pi Ry}{b}, \quad m = m(j_0, \varepsilon), \; n = n(j_0, \varepsilon) \tag{3.6}$$

where the number of degrees of freedom is sufficiently large. In studying the behavior of layered structures with transverse shear taken into account, this number of degrees of freedom increases sharply.

4. In the work above we have identified the fundamental degrees of freedom in systems with distributed parameters [see Eqs. (1.13), (2.6), and (3.6)]. If for a rod the number of degrees of freedom is either small or equal to one, for shells it is fairly large. In the experiments on spherical shells reported in [10, 11], "it was established that the number of bulges in the shells increased with an increase in rate of loading, i.e., much higher forms of stability loss were discovered" ([10], p. 132) and "at much higher rates of loading there was observed a tendency towards the formation of several series of waves, positioned in a checkered fashion concentrically relative to a pole" ([11], p. 39). However, in these experiments a sufficiently stable picture of the bulge formation was not observed. Some "instability," along with a greater sensitivity to the experimental conditions, is connected with the fact that the set of stability loss forms have a tendency towards rapid growth. Normal deflection "pulsations" are possible: "a loss of stability always preceded a considerable or rapidly decaying oscillation of the whole shell or some portion of it (a'splash')," usually "a bulge appeared initially at the contour; with repeated loading this bulge developed further and a second bulge appeared" ([10], pp. 126, 127).

We use the relations (1.13), (2.6), and (3.6) to obtain estimates of the critical buckling time t_* and the critical loading intensity η_*. We first note the features of the aforementioned relationships: each term of the series consists of factors, in the first of which, for t values sufficiently large, the main contribution comes from the exponent, while the second factor does not exceed one in absolute value. Therefore if we assume that the initial perturbations, corresponding to the various forms of the normal motions, are of a single order of smallness [1] [i.e., the relations (1.11) fall into a single class], the following estimate holds for the normal deflection of a shell or a rod:

$$\max |w| \approx KC \exp \alpha^* t \qquad (\alpha^* = \max \alpha) \tag{4.1}$$

Here K is the number of degrees of freedom, and C is a constant depending on the conditions of the problem. The estimate (4.1) holds only for sufficiently large values of t since it contains no terms with $\exp(-\alpha^* t)$ and no terms depending either on x or on t.

The critical buckling time t_* and the critical loading intensity η_* may be obtained from the relationships

$$\max |w(\eta, t_*)| = w_*, \; \max |w(\eta_*, t_0)| = w_* \tag{4.2}$$

if we choose the maximum deflection as the determining quantity.

In Eqs. (4.2) we substitute the estimate (4.1) for the deflection w; we have

$$\alpha^*(\eta)\,t \approx \ln w_* - \ln C - \ln K \tag{4.3}$$

whence the procedure for obtaining the critical parameters t_* and η_* becomes obvious. Equation (4.3) is fairly stable with respect to possible errors (perturbations) in the determination of the constants w_*, C, and K. The term ($-\ln K$), as a rule, has significance in problems involving the buckling of well-defined thin-walled shells, since situations are possible for which $K \gg 1$ [see the examples given in Sec. 2 and 3; in rod-buckling problems this term can, as a rule, be neglected since $K = O(1)$].

LITERATURE CITED

1. M. A. Lavrent'ev and A. Yu. Ishlinskii, "Dynamic forms of stability loss of elastic systems," Dokl. Akad. Nauk SSSR, 64, No. 6 (1949).
2. A. S. Vol'mir, Stability of Deformable Systems [in Russian], Nauka, Moscow (1967).
3. V. M. Kornev, "The forms of stability loss of an elastic rod during impact," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 3 (1968).
4. V. M. Kornev, "The forms of stability loss of elastic shells under intensive loading," Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No. 2 (1969).
5. K. O. Friedrichs, "Asymptotic phenomena in mathematical physics," Bull. Amer. Math. Soc., 61, 485-504 (1955).
6. V. Z. Vlasov, Selected Works, Vol. 1 [in Russian], Academy of Sciences of the USSR, Moscow (1962).
7. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. 2, Wiley-Interscience, New York (1962).
8. V. V. Bolotin, "The density of the frequencies of characteristic oscillations of thin elastic shells," Prikl. Matem. i Mekhan., 27, No. 2 (1963).
9. N. N. Bendich and V. M. Kornev, "The density of the characteristic values in stability problems of thin elastic shells," Prikl. Matem. i Mekhan., 35, No. 2 (1971).
10. R. G. Surkin, B. M. Zuev, and S. G. Stepanov, "Experimental study of the dynamic stability of spherical segments," Prikl. Mekhan., 3, No. 8 (1967).
11. Yu. K. Bivin, "Study of the behavior of cylindrical and spherical shells for short duration loading," in: Transient Deformations of Shells and Plates [in Russian], Tallin (1967).